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Weighted Trigonometric Approximation and Inner–Outer Functions on Higher Dimensional Euclidean Spaces

ROBERT K. GOODRICH AND KARL E. GUSTAFSON

Department of Mathematics, University of Colorado, Boulder, Colorado 80309

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INTRODUCTION

In this paper we consider a class of weighted trigonometric approximation problems in $\mathscr{L}_2(\mathscr{R}^n)$. Many of these questions go back to the classic papers of Beurling [1, 2]. One may describe the general problem as that of finding the best approximation to an arbitrary function from the closed subspace generated by functions of the form $e^{i(v,w)}f(w)$, where f is a fixed function and where the arguments v are to be taken from a given set S. For $\mathscr{L}_2(\mathscr{R}^1)$ and S a half-line, these problems, and their solution, are intimately related to the theory of inner and outer functions. The latter theory does not exist for $\mathscr{R}^n$, $n \geqslant 2$. However, by use of functional analytic and group theoretic methods we are able to treat the trigonometric approximation problem for $\mathscr{L}_2(\mathscr{R}^n)$ with S either a half-space or a quadrant. In so doing, we also arrive at a beginning of a new theory of inner and outer functions for $\mathscr{R}^n$.

The theory of inner and outer functions also plays a central role in the prediction problem for one-parameter processes; see Dym and McKean [4]. In Gustafson and Misra [5] regular one-parameter processes were characterized in terms of the canonical commutation relations of quantum mechanics. Particular emphasis was placed on the use of cyclic vectors. This approach enables us to investigate the weighted trigonometric approximation problems via several-parameter regular processes without appeal to the theory of several complex variables. As a byproduct, we obtain an abstract characterization of the regular representation of $\mathscr{R}^n$. This answers a question raised by Chatterji [3, p. 24]. We also observe a new proof for the existence and uniqueness of outer functions on $\mathscr{R}^1$. This comes out of our (new) formulation and results for inner and outer functions on $\mathscr{R}^n$.

The theory of inner and outer functions for compact abelian groups whose dual is linearly ordered has been developed. See, for example, Helson and

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Lowdenslager [7, 8] and the book by Rudin [16]. For a recent connection between the compact case and several complex variables see Rubel [15].

As pointed out in [5], the use of the canonical commutation relations, or more generally systems of imprimitivity, for regular and other processes, is not new. For example, see the books by Helson [6] and Lax and Phillips [9] and the references therein.

For simplicity all statements in this paper will be given for $\mathscr{R}^2$. Analogous results hold in every case for $\mathscr{R}^n$ under suitable modifications. There are, however, various possible directions to take in higher dimensions, which we have not enumerated here.

In Section 1 we give some preliminaries and descuss the connection with the prediction problem in $\mathscr{L}_2(\mathscr{R}^1)$ and with the factorization problem in the frequency domain. In Section 2 we consider a class of weighted trigonometric approximation problems in higher dimensions. For these, we obtain a general factorization into inner and outer functions in the transform domain. The lack of such factorizations has been a major obstacle in the development of a digital filtering theory in higher dimensions. In Section 3 we give some examples which serve to delineate so-called regular processes from weak regular processes. In Section 4 we indicate briefly the applications to digital filtering in higher dimensions, and in the Appendix we give a short proof of Wiener's theorem on the denseness of a set of translates of a given functions.

1. Preliminaries

First we discuss the one dimensional case, $\mathscr{R}^1$.

Let $\{U_t\}$ be a continuous unitary representation of the real line on a Hilbert space $\mathscr{H}$. We will assume $\{U_t\}$ has a cyclic vector φ, i.e.,

$$\mathscr{H} = \overline{\operatorname{span}}\{U_t(\varphi) \mid -\infty < t < \infty\}, \tag{1.1}$$

where $\overline{\operatorname{span}}\, S$ denotes the closure of the span of a set S in $\mathscr{H}$. Let

$$\mathscr{H}_s = \overline{\operatorname{span}}\{U_t(\varphi) \mid t \leqslant s\}. \tag{1.2}$$

The process $t \to U_t(\varphi)$ is called regular provided that

$$\bigcap_s \mathscr{H}_s = \{0\}. \tag{1.3}$$

Condition (1.3) is sometimes described as the emptyness of the infinitely remote past of the process.

Example 1.1. Let R_t be the regular representation, i.e.,

$$R_t(f)(x) = f(x-t) \tag{1.4}$$

for all $f \in \mathscr{L}_2(\mathscr{R}^1)$. Let φ be a cyclic vector with the support of φ contained in the interval $(-\infty, 0]$. It is clear that condition (1.3) holds since the support of each function in $\mathscr{H}_s$ is contained in $(-\infty, s]$.

Example 1.1 is canonical in the sense that in [5, Theorem 1] it is shown that any regular process is unitarily equivalent to (1.4) with a cyclic vector φ, and with the support of φ contained in $(-\infty, 0]$. In Section 2 we will see that one may obtain a result stronger than that noted in [5]. Namely, any regular process is unitarily equivalent to the regular process with a cyclic vector φ satisfying

$$\overline{\text{span}}\,\{\phi(x-t) \mid t \leqslant 0\} = \mathscr{L}_2(-\infty, 0]. \tag{1.5}$$

This fact follows from a more general result (Corollary 2.1) proven in Section 2. That result for the one dimensional case is known [4] and we may state it here:

COROLLARY 1.1. *If ψ is any nontrivial $\mathscr{L}_2(\mathscr{R}^1)$ function with support contained in $(-\infty, 0]$, then there exists a measurable function g such that $|g| = 1$ almost everywhere and such that $\varphi = \check{}(g\hat{\psi})$ is a cyclic vector for the regular representation satisfying condition* (1.5).

Throughout, $\hat{\psi}$ denotes the Fourier transform of ψ and $\check{}\psi$ denotes the inverse Fourier transform of ψ, see Rudin [17, p. 187].

For a cyclic vector φ of the regular representation R_t satisfying (1.5) one can easily compute the orthogonal projection E_s of $\mathscr{H}$ onto $\mathscr{H}_s$ for the process generated by φ according to the simple rule

$$E_s(f) = \chi_{(-\infty, s]} f \tag{1.6}$$

for all f in $\mathscr{L}_2(\mathscr{R}^1)$, where $\chi_{(-\infty, s]}$ is the characteristic function of $(-\infty, s]$. Thus, given any cyclic vector ψ such that the regularity condition (1.3) is satisfied, one can explicitly (in principle) compute the projections E_s corresponding to the process generated by ψ from the g of Corollary 1.1. Let us emphasize this point, which is in other terms, that in this way one can find for any given f in $\mathscr{L}_2(\mathscr{R}^1)$ the best $\mathscr{L}_2(\mathscr{R}^1)$ approximation to f from functions in $\mathscr{H}_s$, that is, $E_s f$.

This in fact goes to the heart of the so-called prediction problem (see [4]), that is, the computing of the best $\mathscr{L}_2(\mathscr{R}^1)$ approximation to f from negative translates of ψ. For an equivalent formulation of the prediction problem, let us take the Fourier transform of f and ψ. From this we see that the best $\mathscr{L}_2(\mathscr{R}^1)$ approximation to $\hat{f}$ by functions from the subspace $\overline{\text{span}}\,\{e^{ixt}\hat{\psi} \mid t \leqslant s\}$ is given by $\hat{}E_s(f)$. This is the solution (in principle) of the weighted trigonometric approximation problem for $\mathscr{L}_2(\mathscr{R}^1)$ for such ψ.

The calculation of g is not trivial, and it leads one into the theory of inner and outer functions in $\mathscr{H}^2$ (Hardy space). Let us now give a very brief description of this theory, following the development given in [4]. Define the Hardy class $\mathscr{H}^{2+}$ to be the set of functions h which are analytic in the upper half-plane $\mathscr{R}^{2+}$, with norm

$$\|h\|_{2+} = \sup_{b>0} \|h_b\|_2 = \sup_{b>0} \left(\int \left|h(a+ib)\right|^2 da\right)^{1/2} < \infty, \tag{1.7}$$

where $h_b(a) = h(a+ib)$. If $\check{f}(x) = 0$ for $x \leqslant 0$, then

$$h(\omega) = \int_0^\infty e^{i\omega x} \check{f}(x)\, dx \tag{1.8}$$

is of class $\mathscr{H}^{2+}$, $\|h\|_{2+}^2 = \|f\|_2^2$, and

$$\lim_{b\to 0} \|h_b - f\|^2 = 0. \tag{1.9}$$

Conversely, given any $h \in \mathscr{H}^{2+}$, there exists an f in $\mathscr{L}_2$ with $\check{f}(x) = 0$ for all $x \leqslant 0$ and such that (1.8) holds. Because of (1.9) one says that f is the boundary value for h.

The class $\mathscr{H}^{2-}$ is similarly defined as analytic functions on the lower half-plane $\mathscr{R}^{2-}$ with

$$h(\omega) = \int_{-\infty}^0 e^{i\omega x} (\check{f}(x))\, dx \tag{1.10}$$

with $\check{f}(x) = 0$ for $x \geqslant 0$. The two spaces $\mathscr{H}^{2+}$ and $\mathscr{H}^{2-}$ are related since $h^{\#}(w) \equiv \bar{h}(\bar{w})$ is in $\mathscr{H}^{2+}$ if and only if h in $\mathscr{H}^{2-}$.

DEFINITION 1.1. A function $h \in \mathscr{H}^{2+}$ is an outer function if it is nontrivial and

$$\ln|h(\omega)| = \frac{b}{\pi} \int_{-\infty}^\infty \frac{\ln|h_0(\lambda)|}{(\lambda - a)^2 + b^2}\, d\lambda, \tag{1.11}$$

where h_0 is the boundary value of h, and $\omega = a + ib$.

DEFINITION 1.2. A function g is an inner function if it is analytic on $\mathscr{R}^{2+}$, if $|g| < 1$ on $\mathscr{R}^{2+}$, and $|g| = 1$ almost everywhere on $\mathscr{R}^1$.

It is known (see [4, p. 37]) that any nontrivial element of $\mathscr{H}^{2+}$ can be factored as the product of an outer function and an inner function, and this can be done in only one way, up to constant factors of modulus one. The outer factor is given by

$$h^0(\omega) = \exp \frac{1}{\pi i} \int_{-\infty}^\infty \frac{(\lambda\omega + 1)}{(\lambda - \omega)} \frac{\ln|h_0(\lambda)|}{(\lambda^2 + 1)}\, d\lambda. \tag{1.12}$$

Then one may define the inner factor as $g = h/h^0$.

By [4, p. 39], the $\overline{\text{span}}\{e^{i\lambda x}h \mid x \geqslant 0\}$ is $g\mathscr{H}^{2+}$ if g is any inner factor of the nontrivial term h in $\mathscr{H}^{2+}$. In particular

$$\overline{\text{span}}\{e^{i\lambda x}h \mid x \geqslant 0\} = \mathscr{H}^{2+} = \hat{\ }\mathscr{L}_2(0, \infty] \tag{1.13}$$

if and only if h is outer. By the Plancherel theorem h is an outer function if and only if

$$\overline{\text{span}}\{\check{\ }h(t-x) \mid t \leqslant 0\} = \mathscr{L}_2(-\infty, 0]. \tag{1.14}$$

These results readily imply Corollary 1.1. In fact if $\psi \in \mathscr{L}_2(-\infty, 0]$ and nontrivial, then $\hat{\psi} \in \mathscr{H}^{2-}$ and $\hat{\psi}^{\#}$ is in $\mathscr{H}^{2+}$, and moreover $\hat{\psi}^{\#} = h_0 g_1$, where g_1 is an inner function and where h_0 is an outer function. Let $g = (g_1^{\#})^{-1}$. It may be verified that g then satisfies the conclusion of Corollary 1.1.

We see from the above discussion that on the real line, regular processes can be analyzed in terms of inner and outer functions. However, there are problems with extending the theory of inner and outer functions to $\mathscr{R}^n$, because the inner functions employ Blaschke products [5, p. 53]. Let us here quote Zygmund [23, Chap. XVII, p. 316]:

> This case ($m = 1$) is rather exceptional and many results facilitating its study are false when $m > 1$. For example, the zeros of regular functions of a single variable are isolated; and if we divide any f from $\mathscr{H}^p$ or $\mathscr{N}$ by the corresponding Blaschke product we do not alter the class of the function, so that we can thereby reduce the general case to that of a function without zeros. The zeros of regular functions of several variables, on the other hand, form continua and no analogue of the Blaschke product exists, with the result that the theory of classes $\mathscr{H}^p$ and $\mathscr{N}$ is much less complete.

Taking (1.14) as a starting point, we will obtain some results about inner and outer functions in higher dimensions without appeal to complex variable methods. From these we solve a class of weighted trigonometric approximation problems in $\mathscr{L}_2(\mathscr{R}^2)$.

2. The Regular Representation and Two Dimensional Regular Processes

We begin by giving an abstract characterization of the regular representation of $\mathscr{R}^2$, and then define and characterize two dimensional regular processes.

Let $U_{(x,y)}$ be a continuous unitary representation defined on a Hilbert space $\mathscr{H}$ with cyclic vector φ, i.e., $\overline{\text{span}}\{U_{(x,y)}(\varphi) \mid (x, y) \in \mathscr{R}^2\} = \mathscr{H}$. It follows that $\mathscr{H}$ is a separable Hilbert space (recall that a unitary representation is defined to be continuous in the strong operator topology). Let $E_{(s,t)}$, E_s and F_t be the projections of $\mathscr{H}$ onto, respectively,

$$\begin{aligned} &\overline{\text{span}}\{U_{(x,y)}\varphi \mid x \leqslant s,\ y \leqslant t\},\\ &\overline{\text{span}}\{U_{(x,y)}\varphi \mid x \leqslant s\},\\ &\overline{\text{span}}\{U_{(x,y)}\varphi \mid y \leqslant t\}. \end{aligned} \tag{2.1}$$

Denote the range of any projection P by $\mathscr{R}(P)$.

THEOREM 2.1. *A unitary representation $U_{(x,y)}$ of $\mathscr{R}^2$ on a nontrivial Hilbert space $\mathscr{H}$ is unitarily equivalent to the regular representation R of $\mathscr{R}^2$ if and only if there exists a cyclic vector φ with*

$$\bigcap_s \mathscr{R}(E_s) = \{0\} = \bigcap_t \mathscr{R}(F_t). \tag{2.2}$$

Proof. Let U be a unitary representation of $\mathscr{R}^2$ on $\mathscr{H}$ with cyclic vector φ. Notice that $U_{(x,y)}E_sU_{(-x,-y)}$ and E_{s+x} are two orthogonal projections with the same range and hence

$$U_{(x,y)}E_sU_{(-x,-y)} = E_{s+x}, \tag{2.3a}$$

and similarly,

$$U_{(x,y)}F_tU_{(-x,-y)} = F_{t+y}. \tag{2.3b}$$

The family of projections $\{E_s\}$ is monotone in s and hence the limits $E_{s^+} = \lim_{s\to s^+} E_s$ and $E_{s^-} = \lim_{s\to s^-} E_s$ both exist. It is easy to show $E_{s^+} - E_{s^-} = U_{(s,0)}(E_{0^+} - E_{0^-})\, U_{(-s,0)}$ for all s. Thus all of the operators $E_{s^+} - E_{s^-}$ are zero or all of them are nonzero. If $\varphi_s \in \mathscr{R}(E_{s^+} - E_{s^-})$ and $\varphi_s \neq 0$ for all s then the family $\{\varphi_s\}$ forms an uncountable mutually orthogonal family in $\mathscr{H}$, which is impossible in a separable Hilbert space. Thus $E_+ = E_{s^-}$, and similarly $F_{t^+} = F_{t^-}$.

From the addition assumption (2.2) we see that the families $\{E_s\}$ and $\{F_t\}$ generate projection-valued measures. Thus $V_{(0,t)} = \int e^{ixt}\, dF_x$ and $V_{(s,0)} = \int e^{ixs} dE_x$ are representations of $\mathscr{R}^1$. Thus from (2.3) we have the commutation relations

$$\begin{aligned} U_{(x,y)}V_{(0,q)}U_{(-x,-y)} &= \int e^{iqt}\, dU_{(x,y)}F_tU_{(-x,-y)} = \int e^{iqt}\, dF_{(t+y)}\\ &= \int e^{iq(t-y)}\, dF_t = e^{-iqy}V_{(0,q)},\\ U_{(x,y)}V_{(0,q)}U_{(-x,-y)} &= e^{-iqy}V_{(0,\,q)},\\ U_{(x,y)}V_{(r,0)}U_{(-x,-y)} &= e^{-irx}V_{(r,0)}. \end{aligned} \tag{2.4}$$

By Stone's theorem (see Naimark [13, p. 419]), $U_v = \int e^{i(v,w)}\,dP(w)$ for some projection-valued mueasure P on $\mathscr{R}^2$. Let

$$\nu(e) = (P(e)\phi, \phi) \tag{2.5}$$

for all Borel sets e in $\mathscr{R}^2$. Let $T_f = \int f(w)\,dP(w)$ for any bounded measurable function f. Define a unitary operator U from $\mathscr{H}$ onto $\mathscr{L}_2(\mathscr{R}^2, \nu)$ by first defining U on the dense set $\{T_f\varphi \mid f \text{ bounded and measurable}\}$ according to $U(T_f(\varphi)) = f$. It is standard that this mapping is a well-defined isometry of a dense subset of $\mathscr{H}$ onto a dense subset of $\mathscr{L}_2(\mathscr{R}^2, \nu)$, and hence there exists a unique unitary extension U. Moreover, $UT_fU^{-1} = M_f$ for all bounded measurable functions f, where M_f is defined by $M_f(g) = fg$ for all $g \in \mathscr{L}_2(\mathscr{R}^2, \nu)$. Thus $UU_vU^{-1} = M_{f_v}$, where $f_v(w) = e^{i(v,w)}$, and $UP(e)\,U^{-1} = M_{\chi_e}$. We notice ν is nontrivial since $\mathscr{H}$ is nontrivial.

Returning to (2.4), $V_{(0,-q)}U_{(x,y)}V_{(0,q)} = e^{-iqy}U_{(x,y)}$, which gives

$$\begin{aligned}&\int e^{i(v,w)}\,dV_{(0,-q)}P(w)\,V_{(0,q)}\\ &\qquad = e^{-iqy}\int e^{i(u,w)}\,dP(w) = \int e^{i(v,w-(0,q))}\,dP(w).\end{aligned}$$

Thus by the uniqueness of the projection-valued measure in Stone's theorem (or, for example, [16, Theorem 1.3.6, p. 17]), we have

$$V_{(0,-q)}P(e)\,V_{(0,q)} = P(e + (0,q))$$

for all Borel sets e. Similarly,

$$V_{(-r,0)}P(e)\,V_{(r,0)} = P(e + (r,0)),$$

and

$$V_{(-r,0)}V_{(0,-q)}P(e)\,V_{(0,q)}V_{(r,0)} = P(e + (r,q)). \tag{2.6}$$

The following statements are thus all equivalent: (a) $P(e) = 0$, (b) $P(e + (r,q)) = 0$, (c) $T_{\chi_e} = 0$, (d) $T_{\chi(e+(r,q))} = 0$, (e) $\nu(e) = 0$, (f) $\nu(e + (r,q)) = 0$.

Hence $\nu(e) = 0$ if and only if $\nu(e + (r,q)) = 0$, and ν is a nontrivial measure. By a well-known result for quasi-invariant measures (e.g., see Mackey [10, Lemma 3.3, p. 318]), ν is equivalent to the Lebesgue measure on $\mathscr{R}^2$.

Define an isometry V of $\mathscr{L}_2(\mathscr{R}^2, \nu)$ to $\mathscr{L}_2(\mathscr{R}^2, m)$ by $V(f) = f(d\nu/dm)^{1/2}$, where m is the Lebesgue measure on $\mathscr{R}^2$. Notice $VM_{f_v}V^{-1} = M_{f_v}$. Next we compose the above two isometries with the Fourier transform to conclude that $U_v \equiv U_{(x,y)}$ is unitarily equivalent to the regular representation R_v,

$$R_v(f)(w) = f(w - v). \tag{2.7}$$

The converse part of Theorem 2.1, that the regular representation in $\mathscr{R}^2$ always has such a cyclic vector, is shown in Section 3.

The above Theorem 2.1 gives a characterization of the regular representation in terms of a cyclic vector. Any unitary representation of any locally compact second countable group is cyclic, Mackey [12, Theorem A, p. 47], and thus it would be interesting to know if the regular representation of any such abelian group has a characterization in terms of cyclic vectors.

We now define and characterize regular processes on $\mathscr{R}^2$.

DEFINITION 2.1. Let $U_{(x,y)}$ be a continuous unitary representation of $\mathscr{R}^2$ on a Hilbert space H with cyclic vector φ. We say $v \to U_v(\varphi)$ is a regular process if (2.2) holds and

$$E_s F_t = E_{(s,t)} \tag{2.8}$$

for all (s, t) in $\mathscr{R}^2$.

Let us remark that there are many possible definitions for regularity here. See, for example, Chatterji [3], where U is defined to be a regular process provided that U is unitarily equivalent to the regular representation. Chatterji then poses the question of characterizing the latter. We have so characterized such representations for $\mathscr{R}^2$ in our Theorem 2.1, thus answering Chatterji's question for $\mathscr{R}^2$. This characterization, in terms of cyclic vectors, holds as well for $\mathscr{R}^n$.

We also note that in Tjøstheim [21] the multiplicity theory of the regular representation is considered in terms of commutation relations. However, the characterization of the regular representation in terms of cyclic vectors is not considered in [21].

In the following we will also consider a condition (2.13) that is less stringent than (2.8). We should comment here that our main motivation for (2.8) is that it allows us to analyze the projections $E_{(s,t)}$ by showing that they are unitarily equivalent to the projections given by multiplication by $\chi_{(-\infty,s]x(-\infty,t]}$. Thus if one could compute that unitary equivalence, one could find the best $\mathscr{L}_2(\mathscr{R}^2)$ approximation to any function in $\mathscr{L}_2(\mathscr{R}^2)$ by functions in $\overline{\operatorname{span}}\{e^{i(v,w)}\varphi(w) \mid v=(x,y),\ x \leqslant s,\ y \leqslant t\}$, where φ is a given cyclic vector generating a regular process. This, of course, is a class of weighted trigonometric approximation problems in $\mathscr{L}_2(\mathscr{R}^2)$.

Motivated by (1.5) and (1.14) we define outer functions for $\mathscr{R}^2$.

DEFINITION 2.2. A function $\psi \in \mathscr{L}_2(\mathscr{R}^2)$ is an outer function if

$$\overline{\operatorname{span}}\{\check{\ }\psi(v-w) \mid w=(x,y), x \leqslant 0,\ y \leqslant 0\} = \mathscr{L}_2(-\infty, 0]\, x(-\infty, 0]. \tag{2.9}$$

THEOREM 2.2. *Let $v \to U_v(\varphi)$ be a regular process on $\mathscr{H}$ with cyclic vector φ. Then there exists a unitary mapping V of $\mathscr{H}$ onto $\mathscr{L}_2(\mathscr{R}^2)$ such* *that $VU_vV^{-1} = R_v$ is the regular representation and $\hat{\psi}$ is an outer function, where $\psi = V(\varphi)$.*

Proof. From (2.8) and the proof of Theorem 2.1 we see there is no loss of generality in assuming that $\mathscr{H} = \mathscr{L}_2(\mathscr{R}^2)$ and that U_v is given by $U_v(f)(w) = e^{i(v,w)}f(w)$ for all $f \in \mathscr{L}_2(\mathscr{R}^2)$, and that $U_v = \int e^{i(v,w)}\,dP(w)$, where $P(e) = M_{\chi_e}$.

Since $E_sF_t = E_{(s,t)}$ for all s and t, we see that

$$F_tE_s = (E_sF_t)^* = (E_{(s,t)})^* = E_{(s,t)} = E_sF_t.$$

Thus the projections E_s and F_t commute and so by a well-known result for projection-valued measures (e.g., see Mackey [11, Theorem 2.10, p. 203]) we know there exists a unique projection-valued measure E such that

$$E((-\infty, s]\,x(-\infty, t]) = E_{(s,t)} = E_sF_t. \tag{2.10}$$

Since the representations $V_{(0,q)}$ and $V_{(r,0)}$ given in the proof of Theorem 2.1 commute, we may define a representation of $\mathscr{R}^2$ by $V_{(r,q)} = V_{(r,0)}V_{(0,q)}$. By (2.4) we have the commutation relations

$$U_vU_w = e^{-i(v,w)}V_wU_v \tag{2.11}$$

and

$$U_{-v}E(e)\,U_v = E(e+v), \tag{2.12}$$

where (2.12) follows from (2.11) by an argument similar to that in Theorem 2.1.

We are now in a position to apply the Stone–von Neumann theorem [12, Corollary 2, p. 181] or the imprimitivity theorem [12, Theorem 3.10, p. 174, and Corollary 1, p. 180]. We wish to show that the family $\{U_v, V_w\}$ is irreducible; i.e., we show the only operators commuting with (U_v, V_w) are operators of the form cI for some constant c.

Suppose T commutes with U_v and V_w. Let ψ_1 and ψ_2 be vectors in $\mathscr{H}$; then

$$(U_v\psi_1, T^*\psi_2) = (U_vT\psi_1, \psi_2) = \int e^{i(v,w)}\,dv_1(w) = \int e^{i(v,w)}\,dv_2(w),$$

where $v_1(e) = (P(e)\,\psi_1, T^*(\psi_2))$ and $v_2(e) = (P(e)\,T(\psi_1), \psi_2)$. By [16, p. 17] we have $v_1(e) = v_2(e)$ and $P(e)\,T = TP(e)$ for all e. Thus T commutes with M_{χ_e} for all e, and so T commutes with M_f for all bounded measurable functions f. It is known that $\{M_f | f \in \mathscr{L}_\infty\}$ is a maximal abelian algebra in the space of bounded operators on $\mathscr{H}$ [12, p. 98]. Thus $T = T_g$ for some bounded measurable function g.

From (2.6), $V_{-w}P(e)\,V_w = P(e+w)$, or equivalently, $V_{-w}M_{\chi_e}V_w = M_{\chi_{(e+w)}}$. By an elementary measure theory argument this extends to $V_{-w}M_f V_w = M_{f_w}$ for all bounded measurable functions f, where $(f_w)(v)$ denotes the translation $f(v-w)$. But $T = T_g$ commutes with V_w, so $g_w = g$ for all w in $\mathscr{R}^2$, and thus g is equal almost everywhere to a constant. Thus $T = cI$ for some constant c, and $\{V_w, U_v\}$ is irreducible.

By the imprimitivity theorem there exists a unitary mapping V such that $R_v = VU_vV^{-1}$ is the regular representation and $VE(e)\,V^{-1} = M_{\chi_e}$. Letting $\psi = V\varphi$, ψ is then a cyclic vector for the regular representation, and

$$M_{\chi(-\infty,0]x(-\infty,0]}\psi = VE_{(0,0)}V^{-1}\psi = VE_{(0,0)}\varphi = V(\varphi) = \psi,$$

so the support of ψ is in the third quadrant. Lastly, $\hat{\psi}$ is an outer function since

$$\overline{\text{span}}\{R_v(\psi) \mid x \leqslant 0,\ y \leqslant 0,\ v = (x, y)\} = \mathscr{L}_2(-\infty, 0]\,x(-\infty, 0]$$

because $\overline{\text{span}}\{U_v(\varphi) \mid x \leqslant 0,\ y \leqslant 0,\ v = (x, y)\} = R(E_{(0,0)})$ and $VE_{(0,0)}V^{-1} = M_{\chi(-\infty,0]x(-\infty,0]}$. This completes the proof of the theorem.

Before leaving Theorem 2.2, let us comment further on the form of the above isometry V.

Let ψ be any cyclic vector for the regular representation such that $v \to R_v(\psi)$ is a regular process. Theorem 2.2 states that there exists a unitary V such that $VR_vV^{-1} = R_v$, and such that $VE(e)\,V^{-1} = M_{\chi_e}$. The first observation we make is that if V_1 is any other such unitary mapping then $V = cV_1$, where c is a complex scalar with $|c| = 1$. This follows from the fact that $V_1^{-1}V$ is an operator that commutes with $\{R_v\}$ and $\{E(e)\}$ and from the irreducibility of this family of operators.

Next let us apply the Fourier transform. Then R_v is equivalent to $U_v = M_{f_v}$ with $f_v(w) = e^{i(v,w)}$. V is thus seen to be equivalent to an isometry W with the property that $WU_vW^{-1} = U_v$, and so from the above proof we know that $W = M_g$ with $|g| = 1$. From this, we have now proven the following corollary.

COROLLARY 2.1. *Let ψ be any cyclic vector for the regular representation such that $\psi \to R_v(\psi)$ is a regular process. Then there exists a function g such that $|g| = 1$ a.e., and $g\hat{\psi}$ is an outer function. The function g is unique up to a scalar multiple of absolute value one.*

Applying the above proof of this corollary to the one dimensional case, we arrive at a proof of Corollary 1.1. As noted in the previous section, this proof is parallel to that of [5] but observes additional information about the cyclic vector.

Any function ψ satisfying the hypotheses of Corollary 1.1 is cyclic, because $\hat{\psi} \neq 0$ a.e. The latter condition is equivalent to cyclicity for the regular representation of $\mathscr{R}^1$. We shall give a short proof of this for $\mathscr{R}^n$ in the Appendix.

We should also remark here that Helson [6, p. 46] gives yet another functional analytic proof of Corollary 1.1. His idea is based on a cocycle proof similar to that found in the imprimitivity theorem. Helson's proof suggests an alternate approach to the problem in $\mathscr{R}^n$. The cyclic vector approach, however, seems to be technically simpler.

The results of Theorem 2.2 and Corollary 2.1 may be viewed as an application of the Stone–von Neumann theorem (see Putnam [14]). For some interesting generalizations of this theorem see Segal [20]. We note that the existence of outer and inner functions as we have formulated them is thus a consequence of this theory.

Corollary 2.1 may be viewed as a factorization theorem; i.e., if $v \to R_v(\psi)$ is a regular process then $\hat{\psi} = g\psi_1$, where $|g| = 1$ a.e., and ψ_1 is an outer function, and this factorization is unique up to scalar constants of absolute value one. It would be useful to have a complex variable formula for the outer factor similar to (1.12).

If one replaces condition (2.8) with the less stringent assumption

$$E_s F_t = F_t E_s, \tag{2.13}$$

the above theory still holds with minor modifications. Namely, we see that (2.9) should be replaced by

$$\begin{aligned} \overline{\operatorname{span}}\{\check{\psi}(v-w) \mid w=(x,y), x \leqslant 0\} &= L_2(-\infty, 0]\, x(-\infty, \infty), \\ \overline{\operatorname{span}}\{\check{\psi}(v-w) \mid w=(x,y), y \leqslant 0\} &= L_2(-\infty, \infty)\, x(-\infty, 0]. \end{aligned} \tag{2.14}$$

These weaker conditions still allow us to compute the projections E_s and F_t. However, it is not the case that (2.13) implies (2.8), as we shall see in Section 3. It would be of interest to know in what generality (2.13) holds.

3. Some Examples

Example 3.1. If f and g are functions both satisfying (1.5) then $h(x, y) = f(x)\, g(y)$ is a function satisfying (2.2), (2.8) and (2.9). This is so because $\mathscr{R}(E_{(0,0)})$ for h contains all functions of the form $f_1(x)\, g_1(y)$, where the support of f_1 and g_1 is in $(-\infty, 0]$. Thus $\mathscr{R}(E_{(0,0)}) = \mathscr{L}_2(-\infty, 0]\, x(-\infty, 0]$. Hence we have demonstrated the existence of a cyclic vector for the regular representation of $\mathscr{R}^2$ satisfying (2.2), which consequently proves the converse of Theorem 2.1.

For an explicit cyclic vector φ, let $\varphi = f(x)\, g(y)$ with $f = g = \chi_{(-1,0]}$. The functions f and g satisfy (1.5) because $\overline{\operatorname{span}}\{f(x-t) \mid t \leqslant 0\} \subseteq$

$\mathscr{L}_2(-\infty, 0]$, similarly for g, and if $w(x) \in \mathscr{L}_2(-\infty, 0]$, and w is orthogonal to $\overline{\text{span}}\{f(x-t) | t \leqslant 0\}$, then $\int_{t-1}^{t} w(x)\, dx = 0$ for all $t \leqslant 0$. Hence $w(t) = w(t-1)$ for almost all $t \leqslant 0$. But this contradicts $w \in \mathscr{L}_2(-\infty, 0]$ unless $w = 0$.

EXAMPLE 3.2. Let S be the tilted (45°) square in the third quadrant of $\mathscr{R}^2$ with two vertices at $(-1, 0)$, $(0, -1)$. Let $h = \chi_S$. Then h is a cyclic vector for the regular representation satisfying (2.2) and (2.13) but not satisfying (2.8). That h satisfies (2.2) follows easily from the fact that the support of h is in the third quadrant. The range of E_0 is $\mathscr{L}_2(-\infty, 0]\, x(-\infty, \infty)$. To see this, notice that $\mathscr{R}(E_0)$ contains all functions with support between the lines $y = \pm x - t$ and in the left half-plane of $\mathscr{R}^2$, for any real t. This follows from the rotation and translation invariance of Lebesgue measure and Example 3.1. Similarly $\mathscr{R}(F_0)$ is $\mathscr{L}_2(-\infty, \infty) \times (-\infty, 0]$. Thus $E_0 F_0 = F_0 E_0$, which implies $E_s F_t = F_t E_s$ for all s and t. However, $E_0 F_0 \neq E_{(0,0)}$ because the characteristic function of the triangular region bounded by the points $(-1, 0)$, $(0, 0)$ and $(0, -1)$ is a nonzero function in $\mathscr{R}(E_0 F_0)$ that is orthogonal to $\mathscr{R}(E_{(0,0)})$.

The fact that in general $E_{(0,0)}$ and $E_0 F_0$ need not be the same distinguishes the higher dimensional problems from the one dimensional problem, where no such considerations arise.

For example, in higher dimensions there are immediately at least two notions of regular processes. Let $U_{(x,y)}$ be a continuous unitary representation of $\mathscr{R}^2$ on a Hilbert space $\mathscr{H}$ with cyclic vector φ. Let v run through $\mathscr{R}^2$. According to Definition 2.1, $U_v(\varphi)$ is a regular process if (2.2) and (2.8) are satisfied.

DEFINITION 3.1. $U_v(\varphi)$ is a weak regular process if (2.2) and (2.13) hold.

From Example 3.2 we see that the notions of regular process and weak regular process are generally distinct. In like manner, one has a notion of weak outer function (2.14), and a corresponding factorization theorem. Further examples indicate a rich theory in higher dimensions dependent to some extent on the support and geometric properties of φ.

4. APPLICATIONS

Digital filtering theory relates to a number of diverse subjects such as time series analysis, numerical analysis, analog filters, data control systems, econometrics, fast Fourier transform, signal processing, and others, including general stochastic processes and random field theory as seen above. Here we restrict attention to just one aspect of digital filtering theory, where the lack of a factorization method has been a major obstacle to theory and applications in higher dimensions.

For the one dimensional case some indication of these connections may be found in the book by Dym and McKean [4].

An inner function corresponds to what is called in information theory the transfer function of an "all-pass" system. Outer functions correspond (after inverse transform) to input wavelets which are optimal in the sense of producing "maximum gain." A filter is given by the expression

$$Kf = \int \varphi(x-s)\, f(s)\, ds = \langle R_x \varphi, f \rangle, \tag{4.1}$$

where φ has support in the left half-line (past time domain) and where f are arbitrary signal inputs in $L^2(-\infty, \infty)$. K is thus an integral operator generated by a given wavelet φ under regular representation. As is well known, solving (4.1) leads to considerations in Wiener–Hopf theory.

An excellent survey of developments up to 1974 is given in Kailath [24].

Potential applications in higher dimensions include image deblurring and other array processing applications, X-ray enhancement, weather prediction, seismic analysis, among others. For some specific work on 2-D and 3-D digital filtering, see, for example, Mercereau and Dudgeon [25], Ekstrom and Woods [26], Ekstrom and Twogood [27], and the references therein. In each of these papers one will find factorization considerations to be paramount, and a major obstacle to a general theory.

Appendix

Wiener [22] considered the problem of when the translates of an $\mathscr{L}_p(\mathscr{R}^1)$ function are dense in $\mathscr{L}_p(\mathscr{R}^1)$, for $p = 1, 2, \ldots$, in terms of the nonvanishing of the Fourier transform. Segal, in [18, 19], extended these results to arbitrary locally compact abelian groups. For completeness we give here a shorter proof that the translates of a function ψ in $\mathscr{L}_2(\mathscr{R}^2)$ are dense in $\mathscr{L}_2(\mathscr{R}^2)$ if and only if $\hat{\psi}$ vanishes at most on a set of measure zero. Our proof generalizes to locally compact second countable abelian groups, and proves the existence of a cyclic vector for the regular representation.

First suppose $\hat{\psi}$ is zero on a set of positive measure F. Then pick $f \in \mathscr{L}_2(\mathscr{R}^2)$ with support in F. Obviously, f is not in the $\overline{\text{span}}\{e^{i(v,w)}\hat{\psi}(w) \mid v \in \mathscr{R}^2\}$. Taking the inverse Fourier transform we see f is not in the closure of the span of the translates of ψ.

Next suppose $\{w \mid \hat{\psi}(w) = 0\}$ is a set of measure zero. As we saw above, the family of operators $\{M_v\}$, given by $M_v(f)(w) = e^{i(v,w)} f(w)$, generate a maximal commutative family of bounded operators on $\mathscr{L}_2(\mathscr{R}^2)$. As in the proof of Theorem 2.2, the first, and the second, commutant of $\{M_v\}$ is the set of all operators $\{M_g\}$, where $g \in \mathscr{L}_\infty$. By the von Neumann double commutant theorem [13, Corollary 1 and Corollary 2, p. 448] the strong operator closure of the span of $\{M_v\}$ contains all $\{M_g\}$ with $g \in \mathscr{L}_\infty$. Thus span $\{e^{i(v,w)}\hat{\psi}(w) | v \in R^2\} \supseteq \{g\hat{\psi} | g \in \mathscr{L}_\infty\}$. Supposing h is in $\mathscr{L}_2(\mathscr{R}^2)$ and orthogonal to $\{g\hat{\psi} | g \in \mathscr{L}_\infty\}$, letting $g = \chi_e$ for any Borel set e we have $\int_e \bar{h}(w)\,\hat{\psi}(w)\,dw = 0$. By the Radon–Nikodym theorem $\bar{h}(w)\,\hat{\psi}(w) = 0$ a.e., and $h = 0$.

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